DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY

DDC@Cambridge:
FRG Workshop on Definability, Decidability, and Computability

July 29—August 2, 2024
Harvard University Science Center, Room 507

Conference Schedule

Monday, July 29
9 a.m. - 9:30 a.m. Coffee
9:30 a.m. - 10:30 a.m. Bjorn Poonen *Hilbert's tenth problem over rings of integers* **Abstract:** Hilbert's tenth problem asked for an algorithm to decide whether a multivariable polynomial equation has a solution in integers. Davis, Putnam, Robinson, and Matiyasevich proved that there is no such algorithm. I will survey what is known about the analogous question when the integer ring Z is replaced by the ring of integers of a number field.
10:30 a.m. - 11 a.m. Coffee
11 a.m. - 12 p.m. Karen Lange *Integer part inspirations: Complexity of problems related to finding integer parts in real closed fields* **Abstract:** An integer part of a real closed field is a discrete subring in which every element of the field is less than 1 away from an element of the subring. Many years ago Julia Knight and I began work to determine how hard it is to compute an integer part of a given computable real closed field. While this question remains open, our work on integer parts has led us down many interesting paths, from exploring root-taking in generalized power series fields to determining lengths of sums of well ordered sets in ordered groups. Here we'll take a tour through complexity problems related to integer parts and structures relevant to them.
12 p.m. - 2 p.m. Lunch

Monday, July 29

2 p.m. - 3 p.m.

Hector Pasten
An update on Hilbert's tenth problem for complex holomorphic functions

Abstract: One of the main open problems in the context of extensions of Hilbert's tenth problem is that of (transcendental) complex entire functions in one variable. While a lot of progress has been obtained in the algebraic setting, the transcendental case has been more difficult. In this talk I will give an overview of the known results and I will present some recent progress for complex entire functions of bounded growth order.

3:30 p.m. - 4 p.m.

Coffee

4 p.m. - 5:30 p.m.

Students

4 p.m. - 4:30 p.m.
Ian Lewis
Existential definability of big rings over smaller rings
Abstract: In this talk, there will be an overview of Diophantine definability, how it relates to undecidability of analogues of Hilbert's tenth problem and how big rings play a role. A new result will also be presented where a class of subrings of number fields with infinitely many primes in the denominator is defined over a class of smaller rings, which may be chosen with infinitely or finitely many primes in the denominator, using norm equations.

4:40 p.m. -5:20 p.m.
Juan Pablo De Rasis
A universal-existential formula defining affine Campana points in the projective line of a number field
Abstract: Since Matiyasevich's proof of Hilbert's Tenth Problem (no algorithm exists to decide solvability of polynomial equations on the integers), several attempts have been made to expand this to arbitrary rings, rational numbers (and global fields by extension) being among the most interesting cases. If integers were diophantine in the rationals then the result could in fact be extended to the rationals. Motivated on this and previous works by Daans, Dittmann, Eisenträger, Koenigsmann, Miller, Morrison, Park, Poonen, Robinson, Shlapentokh, and others, we present a method to produce a first-order description of a sequence of sets known as Campana points, which constitute a geometrical generalization of n-full integers, and induce a set-theoretical filtration between integers and rationals (and, more generally, between S-integers and scalars of a number field), hoping that this contributes towards approximating Hilbert's Tenth Problem for scalars from the known result in integers.

Tuesday, July 30

9 a.m. - 9:30 a.m.

Coffee

Tuesday, July 30

9:30 a.m. - 10:30 a.m.

Henry Towsner
Fast and faster growing functions

Abstract: The fast-growing hierarchy is a family of functions on the natural numbers, corresponding to "iterating the previous function" over and over again. By extending this iteration process to countable ordinals (more precisely, to make this well-defined, computable presentations of ordinals), this process can be extended to define even faster functions; this means that trying to describe faster growing computable functions largely reduces to describing larger countable ordinals. The Veblen function can be viewed as a way of constructing larger ordinals by a similar iteration (where the successor corresponds to "take fixed-points of the previous function" instead of "iterate the previous function"). We describe how larger ordinals can be seen as repeating the step that took us from functions on the natural numbers to functions on ordinals.

We describe how this process can be used to give a description of the Howard-Bachmann ordinal and then extended to describe the proof-theoretic ordinal of second order arithmetic.

10:30 a.m. - 11 a.m.

Coffee

11 a.m. - 12 p.m.

Natalia Garcia-Fritz
Effectivity for existence of rational points is undecidable

Abstract: After the negative solution of Hilbert's Tenth Problem (H10) for the integers, analogues of this problem for other rings with interesting arithmetic have been extensively studied. The standard technique is to positive existentially define the integers in the new ring, to then translate undecidability. We will focus on the rational numbers. There are almost existential definitions of Z, due to Poonen in 2007 (using two universal quantifiers), and by Koenigsmann in 2010 (using one universal quantifier). In a different direction, in 2003 Poonen proved that there exists a set S of primes with density one such that H10 has a negative answer for $\mathbb{Z}[S^{-1}]$.

In this talk, we will show another approximation to H10 for the rational numbers, by proving that if we add finitely many height comparison conditions for tuples of rational numbers, then the positive existential theory of Q with this new language is undecidable. This is joint work with Hector Pasten and Xavier Vidaux.

12 p.m. - 2 p.m.

Lunch

Tuesday, July 30

2 p.m. - 3 p.m.

Philip Dittmann
Defining subrings of function fields

Abstract: I will consider function fields of curves over some base field K, i.e. the field K(t) and its finite extensions, for a substantial class of base fields K including global fields and complete discretely valued fields. Subrings of such function fields are central examples of "arithmetically interesting subsets" that one may seek to define with first-order formulas. I will report on two instances of this from joint work with Nicolas Daans and Karim Johannes Becher. The first concerns existential definitions of valuation rings, following (as I will explain) a classical theme in research on Hilbert's 10th Problem. I will give some results here powered by local-global principles beyond the classical Hasse principle for quadratic forms, yielding in particular (using established work of Moret-Bailly and Eisenträger-Shlapentokh) new negative results for Hilbert's 10th Problem in certain instances not accessible by previous methods (many of which cannot handle cases where the function field in question contains the algebraic closure of the prime field), but also offering better uniformity properties compared to other approaches.

The second concerns new universal definitions of certain subrings, for instance the polynomial ring Q[t] in the rational function field Q(t) over the rational numbers (but also in considerably greater generality), in rough analogy to definitions of rings of S-integers in global fields (work of Koenigsmann, Park, Eisenträger-Morrison and Daans). Combined with previous work of Miller- Shlapentokh, as an application this yields strong definability results for arbitrary decidable subsets of function fields over number fields.

3:30 p.m. - 4 p.m.

Coffee

Tuesday, July 30

4 p.m. - 5:30 p.m.

Students

4 p.m. - 4:30 p.m.
Keshav Srinivasan
Effective ultrapowers of algebraic number fields
Abstract: We consider a computability-theoretic ultrapower construction for structures. We start with a computable structure, and consider its countable ultrapower over a cohesive set of natural numbers. A cohesive set is an infinite set of natural numbers that is indecomposable with respect to computably enumerable sets. It plays the role of an ultrafilter, and the elements of a cohesive power are the equivalence classes of certain partial computable functions. Thus, unlike many classical ultrapowers, a cohesive power is a countable structure. We focus on the cohesive powers of fields, specifically algebraic number fields, i.e. finite extensions of Q. We analyze the algebraic properties of these cohesive powers, and determine just how far their first-order theory is from the original field, utilizing results of those working on Hilbert's Tenth Problem for Q and related fields.

4:40 p.m. - 5:20 p.m.
Aaron Vollprecht
Characterising a finitely generated algebra over the p-adic integers by a first-order sentence
Abstract: In the beginning of this millenium, Nies investigated the question in which cases groups and rings can be isolated from other groups or rings by a first-order formula. He proved in 2007 that it is possible to express the isomorphism type of the ring of integers within the finitely generated rings by a first-order formula.
In 2020, Aschenbrenner, Khélif, Naziazeno and Scanlon generalised this each finitely generated ring. The proof of this heavily relies on interpretation arguments that were used later by Dittmann and Pop to show an analogous result for finitely generated fields. I will present a version for the class of finitely generated $\mathbb{Z}_p$-algebras and will discuss some of the the difficulties that arise over other base rings.

Wednesday, July 31

9 a.m. - 9:30 a.m.

Coffee

9:30 a.m. - 10:30 a.m.

Adam Topaz
Algebraic dependence is encoded in Milnor K-theory

Abstract: In this talk I'll explain how algebraic dependence is encoded in Milnor K-theory of (almost) completely arbitrary fields. Along the way, I'll mention several motivations for this work, arising from applications in anabelian geometry, field arithmetic and valuation theory.

10:30 a.m. - 11 a.m.

Coffee

Wednesday, July 31

11 a.m. - 12 p.m.

Theodore Slaman
Extending Borel's conjecture from measure to dimension

Abstract: Borel (1919) defined a set of real numbers A to have strong measure zero if for every sequence of positive numbers $(\epsilon_i:i\in\omega)$ there is an open cover of A $(U_i:i\in\omega) $ such that for each i, the diameter of U_i is less than ϵ_i. Besicovitch (1956) showed that A has strong measure zero if and only if A has strong dimension zero, which means that for every gauge function f, A is null for its associated measure H^f. We say that a subset of A of R^n has strong dimension f if and only if $H^f(A)>0$ and for every gauge function g of higher order $H^g(A)=0$. Here, g has higher order than f when $\lim_{t\to0^+}g(t)/f(t)=0.$

Borel conjectured that a set of strong measure zero must be countable. This conjecture naturally extends to the assertion that a set has strong dimension f if and only if it is σ-finite for H^f. Sierpinski (1928) used the continuum hypothesis to give a counterexample to Borel's conjecture and Besicovitch (1963) did the same for its generalization. Laver (1976) showed that Borel's conjecture is relatively consistent with ZFC, the conventional axioms of set theory including the axiom of choice. We will discuss the proof that its generalization to strong dimension is also relatively consistent with $ZFC.$

Free Afternoon

Thursday, August 1

9 a.m. - 9:30 a.m.

Coffee

9:30 a.m. - 10:30 a.m.

Arno Fehm
On the existential theory of the completions of a global field

Abstract: Local-global principles reduce questions about global fields to questions about all (or sometimes almost all) of their completions. I will address the decidability of the question whether a given polynomial equation has a solution in all or almost all completions of a fixed global field. Joint work with Philip Dittmann.

10:30 a.m. - 11 a.m.

Coffee

Thursday, August 1

11 a.m. - 12 p.m.

Franziska Jahnke
AKE principles in mixed characteristic

Abstract: A classical theorem going back to Ax-Kochen and Ershov shows that the theory of an unramified henselian valued field with perfect residue field reduces to those of its residue field and value group. Notably, this gives rise to a model-theoretic understanding of the p-adic numbers. In this talk, we discuss a range of further such transfer principles in valued fields of mixed characteristic, on the one hand generalizing the classical case to finite ramification and imperfect residue fields, and on the other obtaining strengthenings of Kuhlmann's results for tame fields. A special focus lies on the role of the "core field", i.e., the rank-1 step where the characteristic shift happens. The talk contains results obtained in joint work with Anscombe, Dittmann, Kartas, and Ketelsen.

12 p.m. - 2 p.m.

Lunch

2 p.m. - 3 p.m.

Sylvy Anscombe
Universal-existential theories of henselian valued fields, and of function fields

Abstract: Existential theories of equicharacteristic henselian valued fields have been studied quite a bit by now: we have various transfer principles (for decidability of existential theories) --- sometimes conditional, sometimes not. But a comprehensive understanding of the model theory of (for example) power series fields in positive characteristic continues to elude us. In this talk I will describe an investigation of the "universal-existential" (AE) theories of these and related fields, which is joint work with Arno Fehm.

We prove a conditional transfer principle for decidability of the "A1E" theory (i.e. conjunctions and disjunctions of sentences with one universal quantifier followed by an existential formula) of such fields: the A1E theory of a field k is decidable if and only if the A1E theory of k((t)) is decidable. This result is conditional in the same way that earlier existential results were. We also aim to find many-one reductions between other fragments related to the AE fragment. Beyond this, we work on related questions for function fields, and I will discuss to what extent the theory of a field k determines the AE theory of k(t).

3:30 p.m. - 4 p.m.

Coffee

Thursday, August 1

4 p.m. - 5:30 p.m.

Students

4 p.m. - 4:30 p.m.
Andrew Kwon
Diophantine subsets of large fields
Abstract: By work of Fehm and Anscombe, it is known that existentially definable subfields of large fields must be "big". With the additional assumption that the large field is perfect, we show that finite unions of proper subfields cannot be existentially definable. We will provide some background and motivation for large fields and outline the proof.

4:40 p.m. - 5:20 p.m.
Henry Klatt
Quandles are complicated
Abstract: A quandle is a non-associative algebraic structure that naturally arises in knot theory. It is possible to associate to a knot the so-called “fundamental quandle,” which is a complete invariant. In other words, two knots are ‘ambiently isotopic’ if and only if they have isomorphic fundamental quadles. However, this is in some sense an impractical invariant, as quandles are incredibly complicated structures. In this talk, we will demonstrate that the class of (computable) quandles has a maximally complicated isomorphism problem, using tools from computable structure theory.

Friday, August 2

9 a.m. - 9:30 a.m.

Coffee

9:30 a.m. - 12 p.m.

Problem Session